Combining a risky portfolio with a riskless asset

- The riskless asset has a return r_0 and zero variance.
- The risky portfolio Z is a mix of two risky assets with

$$\mu_Z = \lambda_Z \mu_1 + (1 - \lambda_Z)\mu_2 \quad (1)$$

$$\sigma_Z^2 = \lambda_Z^2 \sigma_1^2 + (1 - \lambda_Z)^2 \sigma_2^2 + 2\lambda_Z(1 - \lambda_Z)\sigma_{12} \quad (2)$$

$$\Rightarrow \quad \sigma_Z = \sqrt{\lambda_Z^2 \sigma_1^2 + (1 - \lambda_Z)^2 \sigma_2^2 + 2\lambda_Z(1 - \lambda_Z)\sigma_{12}} \quad (3)$$

  where λ_Z has to be determined in an optimal way. The resulting portfolio Z must lie on the efficiency frontier.
- If portfolio Z is mixed with the riskless asset (share λ_P). The resulting portfolio P has the properties:

$$\mu_P = \lambda_P r_0 + (1 - \lambda_P)\mu_Z \quad (4)$$

$$\sigma_P^2 = (1 - \lambda_P)^2 \sigma_Z^2 \quad (5)$$

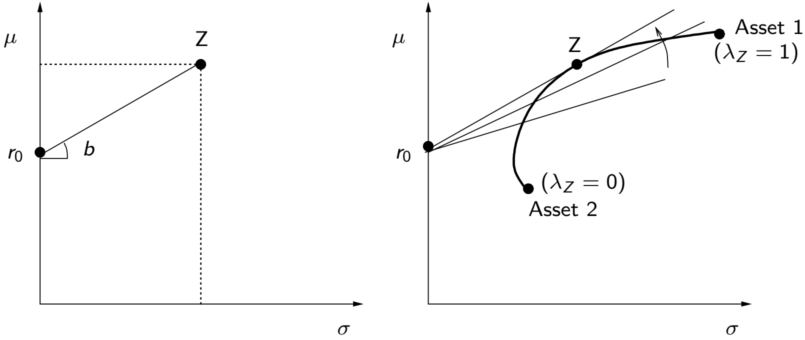
$$\Rightarrow \quad \sigma_P = \sqrt{(1 - \lambda_P)^2 \sigma_Z^2} = (1 - \lambda_P)\sigma_Z \quad (6)$$

  Thus, the risk and return of P is a **linear combination** of Z and the riskless asset.

- In a (σ, μ)-diagram we can therefore write (see also the figure):

$$\mu_P = r_0 + b\sigma_P \tag{7}$$

$$= r_0 + \left(\frac{\mu_Z - r_0}{\sigma_Z}\right)\sigma_P \tag{8}$$



- *How to determine the risky portfolio Z?* Note, that an optimal portfolio P will be a tangential pont of the indifference curve with the linear function (7). Every point on a steeper linear function (7) dominates the points on a flatter linear function since we obtain a higher return with the same standard deviation. Therefore, we maximize the slope $b = (\mu_Z - r_0)/\sigma_Z$ with respect to λ_Z under the condition that μ_Z, σ_Z are defined according to (1) and (3). This guarantees that Z lies on the efficiency frontier. Obviously, Z must be a **tangential** point on the efficiency frintier!

$$\max_{\lambda_Z} b = \frac{\mu_Z - r_0}{\sigma_Z} \quad s.t. \quad (1), (3)$$

$$\Rightarrow \quad \lambda_Z^* = \frac{(\mu_1 - r_0)\sigma_2 - (\mu_2 + r_0)\sigma_{12}}{(\mu_1 - r_0)\sigma_2 + (\mu_2 - r_0)\sigma_1 + 2r_0\sigma_{12} - (\mu_1 + \mu_2)\sigma_{12}}$$

- Now the optimal risky portfolio Z is determined. The linear equation (7) with λ_Z^* (and henceforth b^*) is called **Capital Allocation Line** (CAL).

- Now, the optimal mix between Z and the riskless asset is determined as usual as the tangential point of the indifference curve with the CAL. The solution depends on the degree of risk aversion. But it is remarkable, that irrespective to the individual risk aversion, all rational investors will choose the same risky portfolio Z (**Tobin separation**).